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Level-one Highest Weight Representations of $U_q[gl\widehat{(1|1)}]$ and Associated Vertex Operators

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Abstract

We study the level-one irreducible highest weight representations of $U_q[gl\widehat{(1|1)}]$ and associated q-vertex operators. We obtain the exchange relations satisfied by these vertex operators. The characters and supercharacters associated with these irreducible representations are calculated.

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1 Introduction

This paper is concerned with the level-one irreducible highest weight representations and associated q-vertex operators of the simplest quantum affine superalgebra $U_q[\widehat{gl(1|1)}]$.

Free bosonic realization of level-one representations and the corresponding q-vertex operators [1] of quantum affine (bosonic) algebras have been investigated by a number of groups (see e.g. [2, 3, 4, 5]). Such kind of bosonization construction has been recently extended to the case of type I quantum affine superalgebras $U_q[\widehat{sl(M|N)}],\ M \neq N$ [6] and $U_q[\widehat{gl(N|N)}]$ [7]. However, the level-one irreducible highest weight representations and associated q-vertex operators have been studied for $U_q[\widehat{sl(2|1)}]$ only [6, 8]. As is expected, the representation theory of the super cases is much more complicated than that of the non-super cases.

It is well known by now that infinite dimensional irreducible highest weight representations and associated q-vertex operators play a very powerful role in the algebraic analysis of massive integrable systems (see e.g. [9, 10, 11, 8, 12]). Under some reasonable assumptions on the physical space of states, this algebraic analysis method [9, 10] based on the infinite dimensional non-abelian quantum affine (super)algebra symmetries enables one to compute the correlation functions and form factors of massive (super) integrable systems in the form of integral representations.

In this paper we study in details the level-one irreducible highest weight representations of $U_q[\widehat{gl(1|1)}]$ and associated vertex operators by using the free bosonic realization given in [7]. We calculate the exchange relations satisfied by the vertex operators, and compute the characters and supercharacters associated with these irreducible representations .

2 Bosonization of $U_q[\widehat{gl(1|1)}]$ at Level-One

2.1 Drinfeld basis of $U_q[\widehat{gl(1|1)}]$

The simple roots for $\widehat{gl(1|1)}$ are $\alpha_0 = \delta - \varepsilon_1 + \varepsilon_2$, $\alpha_1 = \varepsilon_1 - \varepsilon_2$ with $\delta,\ \{\varepsilon_1,\ \varepsilon_2\}$ satisfying

$$(\delta, \delta) = (\delta, \varepsilon_k) = 0, \qquad (\varepsilon_k, \varepsilon_{k'}) = (-1)^{k+1}\delta_{kk'}, \quad k, k' = 1, 2. \tag{2.1}$$

The generalized symmetric Cartan matrix of $\widehat{gl(1|1)}$ is degenerate. For the reason which will become clear later in the construction of the vertex operator of $U_q[\widehat{gl(1|1)}]$, we extend the Cartan subalgebra [7] by adding to it the element $\alpha_2 = \varepsilon_1 + \varepsilon_2$. The enlarged Cartan matrix of $\widehat{gl(1|1)}$ has elements $a_{ij} = (\alpha_i, \alpha_j),\ i, j = 0, 1, 2$, so that the Cartan matrix $(a_{ij}),\ i, j = 1, 2$ of $gl(1|1)$ is invertible. Denote by $\mathcal{H}$ the extended Cartan subalgebra and by $\mathcal{H}^*$ the dual of $\mathcal{H}$. Let $\{h_0, h_1, h_2, d\}$ be a basis of $\mathcal{H}$, where d is the usual derivation operator. Let $\{\Lambda_0, \Lambda_1, \Lambda_2, \delta\}$ be the dual basis with Λ_j being fundamental weights. Explicitly [7]

$$\Lambda_2 = \frac{\varepsilon_1 - \varepsilon_2}{2}, \qquad \Lambda_1 = \Lambda_0 + \frac{\varepsilon_1 + \varepsilon_2}{2}, \qquad \Lambda_0. \tag{2.2}$$

The quantum affine superalgebra $U_q[\widehat{gl(1|1)}]$ is a quantum (or q-) deformation of the universal enveloping algebra of $\widehat{gl(1|1)}$ and is generated by the Chevalley generators $\{e_i,\ \ f_i\ \ q^{h_j},\ \ d|i = 0, 1,\ \ j = 0, 1, 2\}$. The $\mathbf{Z}_2$-grading of the Chevalley generators is

$[e_i] = [f_i] = 1,\ i = 0, 1$ and zero otherwise. The defining relations are

$$
\begin{aligned}
&hh' = h'h, \qquad \forall h \in \mathcal{H},\\
&q^{h_j} e_i q^{-h_j} = q^{a_{ij}} e_i, \qquad [d, e_i] = \delta_{i0} e_i,\\
&q^{h_j} f_i q^{-h_j} = q^{-a_{ij}} f_i, \qquad [d, f_i] = -\delta_{i0} f_i,\\
&[e_i, f_{i'}] = \delta_{ii'} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}},\\
&[e_i, e_{i'}] = [f_i, f_{i'}] = 0, \qquad \text{for } \ a_{ii'} = 0,\\
&[[e_0, e_1]_{q^{-1}}, [e_0, e_1]_q] = 0, \quad [[f_0, f_1]_{q^{-1}}, [f_0, f_1]_q] = 0.
\end{aligned} \tag{2.3}
$$

Here and throughout, $[a, b]_x \equiv ab - (-1)^{[a][b]} xba$ and $[a, b] \equiv [a, b]_1$.

$U_q[\widehat{gl(1|1)}]$ is a $\mathbf{Z}_2$-graded quasi-triangular Hopf algebra endowed with the following coproduct Δ, counit ϵ and antipode S:

$$
\begin{aligned}
\Delta(h) &= h \otimes 1 + 1 \otimes h,\\
\Delta(e_i) &= e_i \otimes 1 + q^{h_i} \otimes e_i, \qquad \Delta(f_i) = f_i \otimes q^{-h_i} + 1 \otimes f_i,\\
\epsilon(e_i) &= \epsilon(f_i) = \epsilon(h) = 0,\\
S(e_i) &= -q^{-h_i} e_i, \qquad S(f_i) = -f_i q^{h_i}, \qquad S(h) = -h,
\end{aligned} \tag{2.4}
$$

where $i = 0, 1$ and $h \in \mathcal{H}$.

$U_q[\widehat{gl(1|1)}]$ can also be realized in terms of the Drinfeld generators [13] $\{X^{\pm}_m,\ H^j_n,\ q^{\pm H^j_0},$ $c,\ d | m \in \mathbf{Z},\ n \in \mathbf{Z} - \{0\},\ j = 1, 2\}$. The $\mathbf{Z}_2$-grading of the Drinfeld generators is given by $[X^{\pm}_m] = 1$, for $m \in \mathbf{Z}$ and $[H^j_n] = [H^j_0] = [c] = [d] = 0$ for all $j = 1, 2,\ n \in \mathbf{Z} - \{0\}$. The relations satisfied by the Drinfeld generators read [14, 15, 7]

$$
\begin{aligned}
&[c, a] = [d, H^j_0] = [H^j_0, H^{j'}_n] = 0, \qquad \forall a \in U_q[\widehat{gl(1|1)}]\\
&q^{H^j_0} X^{\pm}_n q^{-H^j_0} = q^{\pm a_{1j}} X^{\pm}_n,\\
&[d, X^{\pm}_n] = n X^{\pm}_n, \quad [d, H^j_n] = n H^j_n,\\
&[H^j_n, H^{j'}_m] = \delta_{n+m,0} \frac{[a_{jj'} n]_q [nc]_q}{n},\\
&[H^j_n, X^{\pm}_m] = \pm \frac{[a_{1j} n]_q}{n} X^{\pm}_{n+m} q^{\mp |n| c/2},\\
&[X^+_n, X^-_m] = \frac{1}{q - q^{-1}} \left(q^{\frac{c}{2}(n-m)} \psi^{+,1}_{n+m} - q^{-\frac{c}{2}(n-m)} \psi^{-,1}_{n+m} \right),\\
&[X^{\pm}_n, X^{\pm}_m] = 0 \quad .
\end{aligned} \tag{2.5}
$$

where $[x]_q = (q^x - q^{-x})/(q - q^{-1})$ and $\psi^{\pm,j}_n$ are related to $H^j_{\pm n}$ by relations

$$
\sum_{n \in \mathbf{Z}} \psi^{\pm,j}_n z^{-n} = q^{\pm H^j_0} \exp\left(\pm (q - q^{-1}) \sum_{n>0} H^j_{\pm n} z^{\mp n} \right). \tag{2.6}
$$

The Chevalley generators are related to the Drinfeld generators by the formulae

$$
\begin{aligned}
&h_i = H^i_0, \quad e_1 = X^+_0, \quad f_1 = X^-_0, \quad e_0 = X^-_1 q^{-H^1_0}, \quad f_0 = -q^{H^1_0} X^+_{-1},\\
&h_{2N} = H^{2N}_0, \quad h_0 = c - H^1_0,
\end{aligned} \tag{2.7}
$$

2.2 Level-one free bosonic realization

In this subsection, we briefly review the bosonization of $U_q[\widehat{gl(1|1)}]$ at level one [7].

Let us introduce bosonic oscillators $\{a_n^j,\ c_n,\ Q_{a^j},\ Q_c|n\in \mathbf{Z}, j=1,2,\}$ which satisfy the commutation relations

$$
\begin{aligned}
&[a_n^i,a_m^j]=(-1)^{i+1}\delta_{ij}\delta_{m+n,0}\frac{[n]_q^2}{n}, \qquad [a_0^i,Q_{a^j}]=\delta_{ij}, \qquad i,j=1,2,\\
&[c_n,c_m]=\delta_{n+m,0}\frac{[n]_q^2}{n}, \qquad [c_0,Q_c]=1.
\end{aligned}
\tag{2.8}
$$

The remaining commutation relations are zero. Corresponding to these bosonic oscillators are the q-deformed free bosonic currents

$$
\begin{aligned}
&H^j(z;\kappa)=Q_{A^j}+A_0^j\ln z-\sum_{n\neq 0}\frac{A_n^j}{[n]_q}q^{\kappa|n|}z^{-n},\\
&c(z)=Q_c+c_0\ln z-\sum_{n\neq 0}\frac{c_n}{[n]_q}z^{-n},\\
&H_\pm^j(z)=\pm(q-q^{-1})\sum_{n>0}A_{\pm n}^j z^{\mp n}\pm A_0^i\ln q,
\end{aligned}
\tag{2.9}
$$

where

$$
\begin{aligned}
&A_n^1=a_n^1+a_n^2, \qquad A_n^2=\frac{q^n+q^{-n}}{2}(a_n^1-a_n^2),\\
&Q_{A^1}=Q_{a^1}-Q_{a^2}, \qquad Q_{A^2}=Q_{a^1}+Q_{a^2}.
\end{aligned}
\tag{2.10}
$$

We introduce the Drinfeld currents or generating functions

$$
X^\pm(z)=\sum_{n\in\mathbf{Z}}X_n^\pm z^{-n-1}\ , \quad \psi^{\pm,j}(z)=\sum_{n\in\mathbf{Z}}\psi_n^{\pm,j}z^{-n}\ , \quad j=1,2,
\tag{2.11}
$$

and the q-differential operator defined by $\partial_z f(z)=\frac{f(qz)-f(q^{-1}z)}{(q-q^{-1})z}$. Then ,

Theorem 1 *([7]): The Drinfeld generators of $U_q[\widehat{gl(1|1)}]$ at level one are realized by the free boson fields as*

$$c=1, \tag{2.12}$$
$$\psi^{\pm,j}(z)=e^{H_\pm^j(z)}, \qquad j=1,2, \tag{2.13}$$
$$X^+(z)=:e^{H^1(z;-\frac{1}{2})}\ e^{c(z)}:, \tag{2.14}$$
$$X^-(z)=:e^{-H^1(z;\frac{1}{2})}\ \partial_z\{e^{-c(z)}\}:. \tag{2.15}$$

2.3 Bosonization of level-one vertex operators

We consider the evaluation representation V_z of $U_q[\widehat{gl(1|1)}]$, where V is a two-dimensional graded vector space with basis vectors $\{v_1,v_2\}$. The $\mathbf{Z}_2$-grading of the basis vectors is chosen to be $[v_j]=\frac{(-1)^j+1}{2}$. Let $e_{j,j'}$ be the 2×2 matrices satisfying $e_{i,j}v_k=\delta_{jk}v_i$. Let V_z^{*S} be the dual module of V_z defined by $\pi_{V^{*S}}(a)=\pi_V(S(a))^{st}$, $\forall a\in U_q[\widehat{gl(1|1)}]$, where st is the supertransposition operation.

In the homogeneous gradation, the Drinfeld generators are represented on V_z by [7]

$$\begin{aligned} H^1_m &= \frac{[m]_q}{m} z^m (e_{1,1}+e_{2,2}),\ H^2_m = -z^m\frac{[2m]_q}{m} q^m e_{2,2},\ H^2_0 = -2e_{2,2} \\ H^1_0 &= e_{1,1}+e_{2,2},\ X^+_m = (qz)^m e_{1,2},\ X^-_m = (qz)^m e_{2,1}, \end{aligned} \tag{2.16}$$

and on V_z^{*S} by

$$\begin{aligned} H^1_m &= -\frac{[m]_q}{m} z^m (e_{1,1}+e_{2,2}),\ H^2_m = z^m\frac{[2m]_q}{m} q^{-m} e_{2,2}\ ,\quad H^2_0 = 2e_{2,2}, \\ H^1_0 &= -e_{1,1}-e_{2,2},\quad X^+_m = q^{-1}(q^{-1}z)^m e_{2,1},\ X^-_m = -q(q^{-1}z)^m e_{1,2}. \end{aligned} \tag{2.17}$$

Now, let $V(\lambda)$ be the highest weight $U_q[\widehat{gl(1|1)}]$-module with the highest weight λ. Consider the following intertwiners of $U_q[\widehat{gl(1|1)}]$-modules [10]:

$$\Phi^{\mu V}_\lambda(z) \;:\; V(\lambda) \longrightarrow V(\mu)\otimes V_z, \tag{2.18}$$

$$\Phi^{\mu V^*}_\lambda(z) \;:\; V(\lambda) \longrightarrow V(\mu)\otimes V_z^{*S}, \tag{2.19}$$

$$\Psi^{V\mu}_\lambda(z) \;:\; V(\lambda) \longrightarrow V_z\otimes V(\mu), \tag{2.20}$$

$$\Psi^{V^*\mu}_\lambda(z) \;:\; V(\lambda) \longrightarrow V_z^{*S}\otimes V(\mu). \tag{2.21}$$

They are intertwiners in the sense that for any $x\in U_q[\widehat{gl(1|1)}]$

$$\Xi(z)\cdot x = \Delta(x)\cdot \Xi(z),\qquad \Xi(z) = \Phi^{\mu V}_\lambda(z),\ \Phi^{\mu V^*}_\lambda(z),\ \Psi^{V\mu}_\lambda(z),\ \Psi^{V^*\mu}_\lambda(z). \tag{2.22}$$

These intertwiners are even operators, that is their gradings are $[\Phi^{\mu V}_\lambda(z)] = [\Phi^{\mu V^*}_\lambda(z)] = [\Psi^{V\mu}_\lambda(z)] = [\Psi^{V^*\mu}_\lambda(z)] = 0$. According to [10], $\Phi^{\mu V}_\lambda(z)$ $\left(\Phi^{\mu V^*}_\lambda(z)\right)$ is called type I (dual) vertex operator and $\Psi^{V\mu}_\lambda(z)$ $\left(\Psi^{V^*\mu}_\lambda(z)\right)$ type II (dual) vertex operator. The vertex operators can be expanded in terms of the basis [10]

$$\begin{aligned} \Phi^{\mu V}_\lambda(z) &= \sum_{j=1}^2 \Phi^{\mu V}_{\lambda,j}(z)\otimes v_j, \qquad \Phi^{\mu V^*}_\lambda(z) = \sum_{j=1}^2 \Phi^{\mu V^*}_{\lambda,j}(z)\otimes v^*_j, \\ \Psi^{V\mu}_\lambda(z) &= \sum_{j=1}^2 v_j\otimes \Psi^{V\mu}_{\lambda,j}(z), \qquad \Psi^{V^*\mu}_\lambda(z) = \sum_{j=1}^2 v^*_j\otimes \Psi^{V^*\mu}_{\lambda,j}(z). \end{aligned} \tag{2.23}$$

The intertwining operators which satisfy (2.22) for any $x\in U_q[\widehat{sl(1|1)}]$ have been constructed in [7]. We extend the construction to $U_q[\widehat{gl(1|1)}]$ by requiring that the vertex operators also obey (2.22) for the element $x = H^2_m$, which extends $U_q[\widehat{sl(1|1)}]$ to $U_q[\widehat{gl(1|1)}]$.

Define the even operators

$$\begin{aligned} \phi(z) &= \sum_{j=1}^2 \phi_j(z)\otimes v_j, \qquad \phi^*(z) = \sum_{j=1}^2 \phi^*_j(z)\otimes v^*_j, \\ \psi(z) &= \sum_{j=1}^2 v_j\otimes \psi_j(z), \qquad \psi^*(z) = \sum_{j=1}^2 v^*_j\otimes \psi^*_j(z). \end{aligned} \tag{2.24}$$

Assuming that $\phi(z)$, $\phi^*(z)$, $\psi(z)$ and $\psi^*(z)$ satisfy (2.22) for any $x\in U_q[\widehat{gl(1|1)}]$ and by using the results in [7] for $U_q[\widehat{sl(1|1)}]$, we find

$$\begin{aligned} &\phi_2(z) =: e^{-H^{*,1}(qz;\frac{1}{2})+H^1(zq^2;\frac{1}{2})} e^{c(qz)} : e^{-\sqrt{-1}\pi c_0}, \qquad -\phi_1(z) = [\phi_2(z)\ ,\ f_1]_{q^{-1}}, \\ &\phi^*_1(z) =: e^{H^{*,1}(qz;\frac{1}{2})} : e^{\sqrt{-1}\pi c_0}, \qquad q\phi^*_2(z) = [\phi^*_1(z)\ , f_1]_q, \\ &\psi_1(z) =: e^{-H^{*,1}(qz;-\frac{1}{2})} : e^{-\sqrt{-1}\pi c_0}\ ,\quad \psi_2(z) = [\psi_1(z)\ , e_1]_q, \\ &\psi^*_2(z) =: e^{H^{*,1}(qz;-\frac{1}{2})-H^1(z;-\frac{1}{2})}\partial_z\{e^{-c(qz)}\} : e^{\sqrt{-1}\pi c_0}, \\ &q^{-1}\psi^*_1(z) = [\psi^*_2(z)\ ,\ e_1]_{q^{-1}}. \end{aligned} \tag{2.25}$$

where

$$H^{*,1}(z;\kappa) = Q^*_{A^1} + A^{*1}_0 \ln z - \sum_{n\neq 0} \frac{A^{*1}_n}{[n]_q} q^{k|n|} z^{-n}, \tag{2.26}$$

$$A^{*1}_n = \frac{1}{q^n + q^{-n}} A^2_n, \quad A^{*2}_n = A^1_n, \quad n \neq 0, \tag{2.27}$$

$$A^{*1}_0 = \frac{1}{2} A^2_0, \quad A^{*2}_0 = \frac{1}{2} A^1_0, \quad Q^*_{A^1} = \frac{1}{2} Q_{A^2}, \quad Q^*_{A^2} = \frac{1}{2} Q_{A^1}. \tag{2.28}$$

Since $\phi(z)$, $\phi^*(z)$, $\psi(z)$ and $\psi^*(z)$ satisfy the same intertwining relations as $\Phi^{\mu V}_\lambda(z)$, $\Phi^{\mu V^*}_\lambda(z)$, $\Psi^{V\mu}_\lambda(z)$ and $\Psi^{V^*\mu}_\lambda(z)$ respectively, we have

Proposition 1 *: The vertex operators $\Phi^{\mu V}_\lambda(z)$, $\Phi^{\mu V^*}_\lambda(z)$, $\Psi^{V\mu}_\lambda(z)$ and $\Psi^{V^*\mu}_\lambda(z)$, if they exist, have the same bosonizaion as the operators $\phi(z)$, $\phi^*(z)$, $\psi(z)$ and $\psi^*(z)$, respectively.*

3 Exchange Relations of the Bosonized Vertex Operators

In this section, we derive the exchange relations of the type I and type II bosonized vertex operators of $U_q[\widehat{gl(1|1)}]$. As expected, these vertex operators satisfy the graded Faddeev-Zamolodchikov algebra.

Let $R(z) \in End(V \otimes V)$ be the R-matrix of $U_q[\widehat{gl(1|1)}]$, defined by

$$R(z)(v_i \otimes v_j) = \sum_{k,l=1}^{2} R^{ij}_{kl}(z) v_k \otimes v_l, \qquad \forall v_i, v_j, v_k, v_l \in V, \tag{3.29}$$

where

$$R^{1,1}_{1,1}(z) = 1, \quad R^{2,2}_{2,2}(z) = \frac{zq^{-1} - q}{zq - q^{-1}}, \quad R^{21}_{12}(z) = \frac{q - q^{-1}}{zq - q^{-1}} \ ,$$
$$R^{12}_{12}(z) = R^{21}_{21}(z) = \frac{z-1}{zq - q^{-1}}, \quad R^{12}_{21}(z) = \frac{(q - q^{-1})z}{zq - q^{-1}} \ , \quad R^{ij}_{kl}(z) = 0, \quad \text{otherwise.}$$

The R-matrix satisfies the graded Yang-Baxter equation on $V \otimes V \otimes V$

$$R_{12}(z)R_{13}(zw)R_{23}(w) = R_{23}(w)R_{13}(zw)R_{12}(z),$$

moreover enjoys : (i) initial condition, $R(1) = P$ with P being the graded permutation operator; (ii) unitarity condition, $R_{12}(\frac{z}{w})R_{21}(\frac{w}{z}) = 1$, where $R_{21}(z) = PR_{12}(z)P$; and (iii) crossing-unitarity,

$$R^{-1,st_1}(z)R(z)^{st_1} = \frac{(z-1)^2}{(q^{-1}z - q)(zq - q^{-1})}.$$

The various supertranspositions of the R-matrix are given by

$$(R^{st_1}(z))^{kl}_{ij} = R^{il}_{kj}(z)(-1)^{[i]([i]+[k])}, \qquad (R^{st_2}(z))^{kl}_{ij} = R^{kj}_{il}(z)(-1)^{[j]([l]+[j])},$$
$$(R^{st_{12}}(z))^{kl}_{ij} = R^{ij}_{kl}(z)(-1)^{([i]+[j])([i]+[j]+[k]+[l])} = R^{ij}_{kl}(z).$$

Now we calculate the exchange relations of the type I and type II bosonic vertex operators of $U_q[\widehat{gl(1|1)}]$ in (2.25). Define

$$\oint dz f(z) = Res(f) = f_{-1}, \qquad \text{for a formal series function } f(z) = \sum_{n\in\mathbf{Z}} f_n z^n.$$

Then, the Chevalley generators of $U_q[\widehat{gl(1|1)}]$ can be expressed by the integrals

$$e_1 = \oint dz X^+(z), \qquad f_1 = \oint dz X^-(z).$$

One can also get the integral expression of the bosonic vertex operators $\phi(z)$, $\phi^*(z)$, $\psi(z)$ and $\phi^*(z)$ from (2.25). Using these integral expressions , we arrive at

Proposition 2 *: The bosonic vertex operators defined in (2.25) satisfy the graded Faddeev-Zamolodchikov algebra*

$$\phi_j(z_2)\phi_i(z_1) = \sum_{k,l=1}^{2} R_{ij}^{kl}(\frac{z_1}{z_2})\phi_k(z_1)\phi_l(z_2)(-1)^{[i][j]}, \tag{3.30}$$

$$\psi_i^*(z_1)\psi_j^*(z_2) = \sum_{k,l=1}^{2} R_{kl}^{ij}(\frac{z_1}{z_2})\psi_l^*(z_2)\psi_k^*(z_1)(-1)^{[i][j]}, \tag{3.31}$$

$$\psi_i^*(z_1)\phi_j(z_2) = \phi_j(z_2)\psi_i^*(z_1)(-1)^{[i][j]}, \tag{3.32}$$

and the following invertibility relations

$$\phi_i(z)\phi_j^*(z) = -q\delta_{ij}\ id.$$

In the derivation of this proposition the fact that $R_{ij}^{kl}(z)(-1)^{[k][l]} = R_{ij}^{kl}(z)(-1)^{[i][j]}$ is helpful.

4 Irreducible Highest Weight $U_q[\widehat{gl(1|1)}]$-modules at Level-One

In this section we study in details the irreducible $U_q[\widehat{gl(1|1)}]$-module structure in the Fock space.

We begin by defining the Fock module. Denote by $F_{\lambda_1,\lambda_2;\lambda_3}$ the bosonic Fock spaces generated by $a^i_{-m}, c_{-m}(m>0)$ over the vector $|\lambda_1,\lambda_2;\lambda_3>$:

$$F_{\lambda_1,\lambda_2;\lambda_3} = \mathbf{C}[a^1_{-1}, a^2_{-1}, a^1_{-2}, a^2_{-2}, \cdots; c_{-1}, c_{-2}, \cdots]|\lambda_1,\lambda_2;\lambda_3>,$$

where

$$|\lambda_1,\lambda_2;\lambda_3>= e^{\sum_{i=1}^2 \lambda_i Q_{a^i} + \lambda_3 Q_c}|0> .$$

The vacuum vector $|0>$ is defined by $a^i_m|0>= c_m|0>= 0$ for $i=1,2$ and $m\geq 0$. Obviously,

$$a^i_m|\lambda_1,\lambda_2;\lambda_3>= 0, \quad c_m|\lambda_1,\lambda_2;\lambda_3>= 0, \quad \text{for } i=1,2 \ \text{ and } \ m>0 \ .$$

To obtain the highest weight vectors of $U_q[\widehat{gl(1|1)}]$, we impose the conditions:

$$e_i|\lambda_1,\lambda_2;\lambda_3>= 0, \quad i=0,1, \text{ and } h_j|\lambda_1,\lambda_2;\lambda_3>= \lambda^j|\lambda_1,\lambda_2;\lambda_3>, \quad j=0,1,2.$$

Solving these equations, we obtain the following classification:

- $(\lambda_1, \lambda_2; \lambda_3) = (\beta-\alpha, -\beta; \alpha)$, where α and β are arbitrary complex numbers. The weight of this vector is $(\lambda^0, \lambda^1, \lambda^2)$=$(1+\alpha, -\alpha, 2\beta-\alpha)$. We have $|(1+\alpha)\Lambda_0 - \alpha\Lambda_1 + (2\beta-\alpha)\beta\Lambda_2> = |\beta-\alpha, -\beta; \alpha>$.

According to this classification, let us introduce the Fock spaces

$$\mathcal{F}_{(\alpha;\beta)} = \oplus_{i\in \mathbf{Z}} F_{\beta-\alpha+i,\ -\beta-i;\ \alpha+i} \quad . \tag{4.1}$$

It can be shown that the bosonized action of $U_q[gl\widehat{(1|1)}]$ on $\mathcal{F}_{(\alpha;\beta)}$ is closed :$U_q[gl\widehat{(1|1)}]\mathcal{F}_{(\alpha;\beta)} = \mathcal{F}_{(\alpha;\beta)}$. Hence each Fack space (4.1) constitutes a $U_q[gl\widehat{(1|1)}]$-module. However, these modules are not irreducible in general. To obtain the irreducible representations , we introduce a pair of fermionic currents [6, 8]

$$\eta(z) = \sum_{n\in\mathbf{Z}} \eta_n z^{-n-1} =: e^{c(z)} : \;\; , \quad \xi(z) = \sum_{n\in\mathbf{Z}} \xi_n z^{-n} =: e^{-c(z)} : .$$

The mode expansion of $\eta(z)$ and $\xi(z)$ is well defined on $\mathcal{F}_{(\alpha;\beta)}$ with $\alpha \in \mathbf{Z}$, and the modes satisfy the relations

$$\xi_m\xi_n + \xi_n\xi_m = \eta_m\eta_n + \eta_n\eta_m = 0 \;, \quad \xi_m\eta_n + \eta_n\xi_m = \delta_{m+n,0} \quad . \tag{4.2}$$

Therefore, $\eta_0\xi_0$ and $\xi_0\eta_0$ qualify as projectors . So we use them to decompose $\mathcal{F}_{(\alpha;\beta)}$ into a direct sum $\mathcal{F}_{(\alpha;\beta)} = \eta_0\xi_0\mathcal{F}_{(\alpha;\beta)} \oplus \xi_0\eta_0\mathcal{F}_{(\alpha;\beta)}$. Following [6], $\eta_0\xi_0\mathcal{F}_{(\alpha;\beta)}$ is referred to as Ker_{η_0} and $\xi_0\eta_0\mathcal{F}_{(\alpha;\beta)} = \mathcal{F}_{(\alpha;\beta)}/\eta_0\xi_0\mathcal{F}_{(\alpha;\beta)}$ as $Coker_{\eta_0}$. Since η_0 commutes (anti-commutes) with $U_q[gl\widehat{(1|1)}]$, Ker_{η_0} and $Coker_{\eta_0}$ are both $U_q[gl\widehat{(1|1)}]$-modules.

4.1 Characters and supercharacters

In this subsection, we study the character and supercharacter formulas of these $U_q[gl\widehat{(1|1)}]$-modules which are constructed in the bosonic Fock spaces. We first of all bosonize the derivation operator d as

$$d = -\sum_{1\leq m} \frac{m^2}{[m]_q[2m]_q}\{A^1_{-m}A^2_m + A^2_{-m}A^1_m + \frac{[2m]_q}{[m]_q}c_{-m}c_m\} - \frac{1}{2}\{A^1_0A^2_0 + c_0(c_0+1)\}.$$

One can check that this d obeys the commutation relations

$$[d, h_i] = 0, \qquad [d, h^i_m] = mh^i_m, \qquad [d, X^{\pm}_m] = mX^{\pm}_m, \quad i = 1, 2,$$

as required. Moreover, we have $[d, \xi_0] = [d, \eta_0] = 0$.

The character and supercharacter of a $U_q[gl\widehat{(1|1)}]$-module M are defined by

$$Ch_M(q, x, y) = tr_M(q^{-d}x^{h_1}y^{h_2}) \;, \tag{4.3}$$
$$Sch_M(q, x, y) = Str_M(q^{-d}x^{h_1}y^{h_2}) = tr_M((-1)^{N_f}q^{-d}x^{h_1}y^{h_2}), \tag{4.4}$$

respectively. The Fermi-number operator N_f can be also bosonized by $N_f = c_0$. Indeed, such a bosonized operator satisfies

$$(-1)^{N_f}\Xi(z) = (-1)^{[\Xi(z)]}\Xi(z), \text{ for } \Xi(z) = X^{\pm,i}(z), \phi_i(z), \phi^*_i, \psi_i(z), \psi^*_i(z),$$

as required. Then we have the following result:

- (I) *Character of* $\mathcal{F}_{(\alpha;\beta)}$ *for* $\alpha \notin \mathbf{Z}$. Since η_0 is not defined on this module, it is expected that $\mathcal{F}_{(\alpha;\beta)}$ is actually irreducible. We thus have

Proposition 3 *: The character and supercharacter of* $\mathcal{F}_{(\alpha;\beta)}$ *are*

$$\begin{aligned} Ch_{\mathcal{F}_{(\alpha;\beta)}}(q,x,y) &= \frac{q^{\frac{1}{2}\alpha(2\alpha-2\beta+1)}}{\prod_{n=1}^{\infty}(1-q^n)^3}\sum_{i\in\mathbf{Z}} q^{\frac{1}{2}(i^2+i)}x^{-\alpha}y^{2\beta-\alpha+2i}, \\ Sch_{\mathcal{F}_{(\alpha;\beta)}}(q,x,y) &= \frac{q^{\frac{1}{2}\alpha(2\alpha-2\beta+1)}}{\prod_{n=1}^{\infty}(1-q^n)^3}\sum_{i\in\mathbf{Z}}(-1)^{\alpha+i} q^{\frac{1}{2}(i^2+i)}x^{-\alpha}y^{2\beta-\alpha+2i}. \end{aligned} \tag{4.5}$$

- (II) *Characters and supercharacters of* $Ker_{\mathcal{F}_{(\alpha;\beta)}}$ *and* $Coker_{\mathcal{F}_{(\alpha;\beta)}}$*, for* $\alpha \in \mathbf{Z}$. In this case, η_0 is well defined on $Ker_{\mathcal{F}_{(\alpha;\beta)}}$ and $Coker_{\mathcal{F}_{(\alpha;\beta)}}$. So we compute the characters and supercharacters of these modules by using the BRST resolution [8].

Let us define the Fock spaces, for $l \in \mathbf{Z}$

$$\mathcal{F}^{(l)}_{(\alpha;\beta)} = \oplus_{i\in\mathbf{Z}} F_{\beta-\alpha+i,\ -\beta-i;\ \alpha+i+l}.$$

We have $\mathcal{F}^{(0)}_{(\alpha;\beta)} = \mathcal{F}_{(\alpha;\beta)}$. It can be shown that η_0 intertwine these Fock spaces as follows:

$$\begin{aligned} \eta_0 :\ \mathcal{F}^{(l)}_{(\alpha;\beta)} &\longrightarrow \mathcal{F}^{(l+1)}_{(\alpha;\beta)}\ , \\ \xi_0 :\ \mathcal{F}^{(l)}_{(\alpha;\beta)} &\longrightarrow \mathcal{F}^{(l-1)}_{(\alpha;\beta)}\ . \end{aligned}$$

We have the following BRST complexes:

$$\begin{array}{ccccccc} \cdots & \xrightarrow{Q_{l-1}=\eta_0} & \mathcal{F}^{(l)}_{(\alpha;\beta)} & \xrightarrow{Q_l=\eta_0} & \mathcal{F}^{(l+1)}_{(\alpha;\beta)} & \xrightarrow{Q_{l+1}=\eta_0} & \cdots \\ & & |\mathbf{O} & & |\mathbf{O} & & \\ \cdots & \xrightarrow{Q_{l-1}=\eta_0} & \mathcal{F}^{(l)}_{(\alpha;\beta)} & \xrightarrow{Q_l=\eta_0} & \mathcal{F}^{(l+1)}_{(\alpha;\beta)} & \xrightarrow{Q_{l+1}=\eta_0} & \cdots \end{array} \tag{4.6}$$

where $\mathbf{O}$ is an operator such that $\mathcal{F}^{(l)}_{(\alpha;\beta)} \longrightarrow \mathcal{F}^{(l)}_{(\alpha;\beta)}$. We can get

Proposition 4 *:*

$$\begin{aligned} &Ker_{Q_l} = Im_{Q_{l-1}}, \quad \text{for any } l \in \mathbf{Z}, \quad \text{and} \\ &tr(\mathbf{O})|_{Ker_{Q_l}} = tr(\mathbf{O})|_{Im_{Q_{l-1}}} = tr(\mathbf{O})|_{Coker_{Q_{l-1}}}\ . \end{aligned} \tag{4.7}$$

Proof. It follows from the fact that $\eta_0\xi_0 + \xi_0\eta_0 = 1$, $(\eta_0)^2 = (\xi_0)^2 = 0$ and $\eta_0\xi_0$ $(\xi_0\eta_0)$ is the projection operator from $\mathcal{F}^{(l)}_{(\alpha;\beta)}$ to Ker_{Q_l} $(Coker_{Q_l})$.

In the following we simply write Ker_{η_0} and $Coker_{\eta_0}$ of $\mathcal{F}_{(\alpha;\beta)}$ as $Ker_{\mathcal{F}_{(\alpha;\beta)}}$ and $Coker_{\mathcal{F}_{(\alpha;\beta)}}$, respectively. By proposition 4, we can compute the characters and supercharacters of $Ker_{\mathcal{F}_{(\alpha;\beta)}}$ and $Coker_{\mathcal{F}_{(\alpha;\beta)}}$, for $\alpha \in \mathbf{Z}$. We have

Proposition 5 *: The character and supercharacter of* $Ker_{\mathcal{F}_{(\alpha;\beta)}}$ *and* $Coker_{\mathcal{F}_{(\alpha;\beta)}}$ *for* $\alpha \in \mathbf{Z}$ *are given by*

$$\begin{aligned} Ch_{Ker_{\mathcal{F}_{(\alpha;\beta)}}}(q,x,y) &= \frac{q^{\frac{1}{2}\alpha(2\alpha-2\beta+1)}}{\prod_{n=1}^{\infty}(1-q^n)^3}\sum_{l=1}^{\infty}(-1)^{l+1}q^{\frac{1}{2}(l^2-(1+2\alpha)l)}\sum_{i\in\mathbf{Z}} q^{\frac{1}{2}(i^2+(1-2l)i)}x^{-\alpha}y^{2\beta-\alpha+2i}, \\ Ch_{Coker_{\mathcal{F}_{(\alpha;\beta)}}}(q,x,y) &= \frac{q^{\frac{1}{2}\alpha(2\alpha-2\beta+1)}}{\prod_{n=1}^{\infty}(1-q^n)^3}\sum_{l=1}^{\infty}(-1)^{l+1}q^{\frac{1}{2}(l^2+(1+2\alpha)l)}\sum_{i\in\mathbf{Z}} q^{\frac{1}{2}(i^2+(1+2l)i)}x^{-\alpha}y^{2\beta-\alpha+2i}, \end{aligned} \tag{4.8}$$

and

$$Sch_{Ker_{\mathcal{F}_{(\alpha;\beta)}}}(q,x,y) = -\frac{q^{\frac{1}{2}\alpha(2\alpha-2\beta+1)}}{\prod_{n=1}^{\infty}(1-q^n)^3}\sum_{l=1}^{\infty} q^{\frac{1}{2}(l^2-(1+2\alpha)l)}\sum_{i\in\mathbf{Z}}(-1)^i q^{\frac{1}{2}(i^2+(1-2l)i)}(-x)^{-\alpha}y^{2\beta-\alpha+2i},$$
$$Sch_{Coker_{\mathcal{F}_{(\alpha;\beta)}}}(q,x,y) = -\frac{q^{\frac{1}{2}\alpha(2\alpha-2\beta+1)}}{\prod_{n=1}^{\infty}(1-q^n)^3}\sum_{l=1}^{\infty} q^{\frac{1}{2}(l^2+(1+2\alpha)l)}\sum_{i\in\mathbf{Z}}(-1)^i q^{\frac{1}{2}(i^2+(1+2l)i)}(-x)^{-\alpha}y^{2\beta-\alpha+2i}. \tag{4.9}$$

Proof. Thanks to the resolution of BRST complexes (4.6) , the trace over Ker and $Coker$ can be written as the sum of trace over $\mathcal{F}^{(l)}_{(\alpha;\beta)}$. The latter can be computed by the technique introduced in [16].

Note that $\mathcal{F}^{(1)}_{(\alpha;\beta)} = \mathcal{F}_{(\alpha;\beta-1)}$, we have

Corollary 1 *: The following relations hold for any $\alpha \in \mathbf{Z}$ and β,*

$$Ch_{Coker_{\mathcal{F}_{(\alpha;\beta+1)}}} = Ch_{Ker_{\mathcal{F}_{(\alpha,\beta)}}}\ , \tag{4.10}$$
$$Sch_{Coker_{\mathcal{F}_{(\alpha;\beta+1)}}} = Sch_{Ker_{\mathcal{F}_{(\alpha,\beta)}}}\ . \tag{4.11}$$

4.2 $U_q[\widehat{gl(1|1)}]$-module structure of $\mathcal{F}_{(\alpha;\beta+\frac{3}{2}\alpha)}$

It is easy to see that the vector

$$|\beta+\frac{\alpha}{2},-\beta-\frac{3}{2}\alpha;\alpha> \in \mathcal{F}_{(\alpha;\beta+\frac{3}{2}\alpha)}$$

plays the role of the highest weight vectors of $U_q[\widehat{gl(1|1)}]$-modules. We can also check that

$$\eta_0|\beta+\frac{\alpha}{2},-\beta-\frac{3}{2}\alpha;\alpha> = 0, \quad \text{for } \alpha = 0,1,2,3,\cdots, \tag{4.12}$$
$$\eta_0|\beta+\frac{\alpha}{2},-\beta-\frac{3}{2}\alpha;\alpha> \neq 0, \quad \text{for } \alpha = -1,-2,-3,\cdots. \tag{4.13}$$

It follows that the modules $Ker_{\mathcal{F}_{(\alpha;\beta+\frac{3}{2}\alpha)}}$ $(\alpha = 0,1,2,3,\cdots)$ and $Coker_{\mathcal{F}_{(\alpha;\beta+\frac{3}{2}\alpha)}}$ $(\alpha = -1,-2,-3,\cdots)$ are highest weight $U_q[\widehat{gl(1|1)}]$-modules. Set

$$\lambda_{\alpha,\beta} = \begin{cases} (1+\alpha)\Lambda_0 - \alpha\Lambda_1 + (2\beta+2\alpha)\Lambda_2 & \text{for } \alpha \notin \mathbf{Z} \\ (1+\alpha)\Lambda_0 - \alpha\Lambda_1 + (2\beta+2\alpha)\Lambda_2 & \text{for } \alpha = 0,1,2,3,\cdots \\ (1+\alpha)\Lambda_0 - \alpha\Lambda_1 + (2\beta+2\alpha+2)\Lambda_2 & \text{for } \alpha = -1,-2,-3,\cdots \end{cases}. \tag{4.14}$$

Denote by $\overline{V}(\lambda_{\alpha,\beta})$ the highest weight $U_q[\widehat{gl(1|1)}]$-modules with the highest weights $\lambda_{\alpha,\beta}$. From (4.12)-(4.13) and corollary 1, we obtain

Theorem 2 *: We have the following identifications of the highest weight $U_q[\widehat{gl(1|1)}]$-modules:*

$$\begin{aligned} \overline{V}(\lambda_{\alpha;\beta}) &\cong Ker_{\mathcal{F}_{(\alpha;\beta+\frac{3}{2}\alpha)}} \equiv Coker_{\mathcal{F}_{(\alpha;\beta+\frac{3}{2}\alpha+1)}}, \quad \text{for } \alpha \in \mathbf{Z} \\ &\cong \mathcal{F}_{(\alpha;\beta+\frac{3}{2}\alpha)}, \quad \text{for } \alpha \notin \mathbf{Z} \end{aligned} \tag{4.15}$$

and when $\alpha \in \mathbf{Z}$ each Fock space $\mathcal{F}_{(\alpha;\beta+\frac{3}{2}\alpha)}$ can also be decomposed explicitly into a direct sum of the highest weight $U_q[\widehat{gl(1|1)}]$-modules:

$$\mathcal{F}_{(\alpha;\beta+\frac{3}{2}\alpha)} = \overline{V}(\lambda_{\alpha,\beta}) \oplus \overline{V}(\lambda_{\alpha,\beta-1}), \quad \text{for } \alpha \in \mathbf{Z}. \tag{4.16}$$

It is expected that the modules $\overline{V}(\lambda_{\alpha,\beta})$ are also irreducible with respect to the action of $U_q[gl\widehat{(1|1)}]$. We thus state

Conjecture 1: *$\overline{V}(\lambda_{\alpha,\beta})$ are the irreducible highest weight $U_q[gl\widehat{(1|1)}]$-modules with the highest weight $\lambda_{\alpha,\beta}$, i.e*

$$\overline{V}(\lambda_{\alpha,\beta}) = V(\lambda_{\alpha,\beta}), \tag{4.17}$$

where $V(\lambda)$ denotes the irreducible highest weight $U_q[gl\widehat{(1|1)}]$-module with the highest weight λ.

4.3 Vertex operators over the irreducible highest weight $U_q[gl\widehat{(1|1)}]$-modules

In this subsection we study the action of type I and type II vertex operators of $U_q[gl\widehat{(1|1)}]$ on the irreducible highest weight $U_q[gl\widehat{(1|1)}]$-modules.

Using the bosonic representations of the vertex operators (2.25), we have the homomorphisms of $U_q[gl\widehat{(2|2)}]$-modules:

$$\phi(z): \ \mathcal{F}_{(\alpha;\beta)} \longrightarrow \mathcal{F}_{(\alpha+1;\beta)} \otimes V_z \ , \quad \psi(z): \ \mathcal{F}_{(\alpha;\beta)} \longrightarrow V_z \otimes \mathcal{F}_{(\alpha+1;\beta)} \ , \tag{4.18}$$

$$\phi^*(z): \ \mathcal{F}_{(\alpha;\beta)} \longrightarrow \mathcal{F}_{(\alpha-1;\beta)} \otimes V_z^{*S} \ , \quad \psi^*(z): \ \mathcal{F}_{(\alpha;\beta)} \longrightarrow V_z^{*S} \otimes \mathcal{F}_{(\alpha-1;\beta)} \ . \tag{4.19}$$

We then consider the vertex operators which intertwine the irreducible highest weight $U_q[gl\widehat{(1|1)}]$-modules. By conjecture 1, we can make the fellowing identifications:

$$\Phi_i(z) = \begin{cases} \phi_i(z) & \text{for} \ \ \alpha \notin \mathbf{Z} \\ \eta_0\xi_0\phi_i(z)\eta_0\xi_0 & \text{for} \ \ \alpha \in \mathbf{Z} \end{cases} , \quad \Phi_i^*(z) = \begin{cases} \phi_i^*(z) & \text{for} \ \ \alpha \notin \mathbf{Z} \\ \eta_0\xi_0\phi_i^*(z)\eta_0\xi_0 & \text{for} \ \ \alpha \in \mathbf{Z} \end{cases} ,$$
$$\Psi_i(z) = \begin{cases} \psi_i(z) & \text{for} \ \ \alpha \notin \mathbf{Z} \\ \eta_0\xi_0\psi_i(z)\eta_0\xi_0 & \text{for} \ \ \alpha \in \mathbf{Z} \end{cases} , \quad \Psi_i^*(z) = \begin{cases} \psi_i^*(z) & \text{for} \ \ \alpha \notin \mathbf{Z} \\ \eta_0\xi_0\psi_i^*(z)\eta_0\xi_0 & \text{for} \ \ \alpha \in \mathbf{Z} \end{cases} . \tag{4.20}$$

This implies that the following vertex operators associated with the level-one irreducible highest weight $U_q[gl\widehat{(1|1)}]$-modules exist:

$$\begin{aligned} &\Phi(z)_{\lambda_{\alpha,\beta}}^{\lambda_{\alpha+1,\beta-3/2} \ V}(z): \quad V(\lambda_{\alpha,\beta}) \longrightarrow V(\lambda_{\alpha+1,\beta-3/2}) \otimes V_z, \\ &\Psi(z)_{\lambda_{\alpha,\beta}}^{V \ \lambda_{\alpha+1;\beta-3/2}}: \quad V(\lambda_{\alpha,\beta}) \longrightarrow V_z \otimes V(\lambda_{\alpha+1,\beta-3/2}), \\ &\Phi(z)_{\lambda_{\alpha,\beta}}^{\lambda_{\alpha-1,\beta+3/2} \ V^*}(z): \quad V(\lambda_{\alpha,\beta}) \longrightarrow V(\lambda_{\alpha-1,\beta+3/2}) \otimes V_z^{*S}, \\ &\Psi(z)_{\lambda_{\alpha,\beta}}^{V^* \ \lambda_{\alpha-1;\beta+3/2}}: \quad V(\lambda_{\alpha,\beta}) \longrightarrow V_z^{*S} \otimes V(\lambda_{\alpha-1,\beta+3/2}). \end{aligned} \tag{4.21}$$

Moreover, the vertex operators defined by (4.20) satisfy the graded Faddeev-Zamolodchikov algbera (3.30)-(3.32):

$$\Phi_j(z_2)\Phi_i(z_1) = \sum_{k,l=1}^{2} R_{ij}^{kl}(\frac{z_1}{z_2})\Phi_k(z_1)\Phi_l(z_2)(-1)^{[i][j]} \ , \tag{4.22}$$

$$\Psi_i^*(z_1)\Psi_j^*(z_2) = \sum_{k,l=1}^{2} R_{kl}^{ij}(\frac{z_1}{z_2})\Psi_l^*(z_2)\Psi_k^*(z_1)(-1)^{[i][j]} \ , \tag{4.23}$$

$$\Psi_i^*(z_1)\Phi_j(z_2) = \Phi_j(z_2)\Psi_i^*(z_1)(-1)^{[i][j]}, \tag{4.24}$$

and the following invertibility relations:

$$\Phi_i(z)\Phi_j^*(z)|_{V(\lambda_{\alpha,\beta})} = -q\delta_{ij}\ id|_{V(\lambda_{\alpha,\beta})}. \tag{4.25}$$

We can also generalize Miki's construction to the $U_q[\widehat{gl(1|1)}]$ case. Define

$$L^+(z)_i^j = \Phi_i(zq^{1/2})\Psi_j^*(zq^{-1/2}) \quad ,$$
$$L^-(z)_i^j = \Phi_i(zq^{-1/2})\Psi_j^*(zq^{1/2}) \quad .$$

We have

Proposition 6 *: The L-operators $L^{\pm}(z)$ defined above give a realization of the super Reshetikhin-Semenov-Tian-Shansky algebra [15] at level one for the quantum affine superalgebra $U_q[\widehat{gl(1|1)}]$*

$$R(z/w)L_1^{\pm}(z)L_2^{\pm}(w) = L_2^{\pm}(w)L_1^{\pm}(z)R(z/w),$$
$$R(z^+/w^-)L_1^{+}(z)L_2^{-}(w) = L_2^{-}(w)L_1^{+}(z)R(z^-/w^+),$$

where $L_1^{\pm}(z) = L^{\pm}(z) \otimes 1$, $L_2^{\pm}(z) = 1 \otimes L^{\pm}(z)$ and $z^{\pm} = zq^{\pm\frac{1}{2}}$.

Proof. Straightforward by using (4.22)-(4.24).

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